COMBINATIONS

Now we discuss the lottery problem mentioned in the Introduction of the chapter *Factorials*. There are 51 numbers — you choose 6 of them. The order in which you choose the numbers does not make any difference, as long as they're the same 6 numbers selected on the Big Spin. So, to figure the probability of getting all 6 numbers right, we have to ask:

How many ways are there to select 6 numbers from a set of 51 numbers?

Let's put our toes in the water by solving a simpler problem first.

❑ *COMBINATIONS*

<u>EXAMPLE 1:</u> **How many ways are there to choose 3 letters from the set of 5 letters {a, b, c, d, e}?**

<u>Solution:</u> Remember, the order doesn't count, so the selection "bce," for instance, is the same as "ecb." A little trial and error yields the following list:

abc, abd, abe, acd, ace, ade, bcd, bce, bde, cde

Count them; there are **10 ways** to select 3 letters from the 5.

Can we solve the lottery problem (choosing 6 out of 51) using the same method we just used in Example 1? Sure we could, requiring a boxful of paper and months of writing. There's got to be a formula for this!

Our formula will use the notation $\binom{n}{k}$, read "***n* choose *k***." It's the number of ways we can choose ***k*** objects from a set of ***n*** objects. The formula, using our friend the factorial, is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

The "*n* choose *k*" Formula

$\binom{n}{k}$ is also called "a ***combination*** of ***n*** things taken ***k*** at a time."

Another common notation for $\binom{n}{k}$ is ${}_nC_k$.

Let's redo Example 1 using our new combination formula.

EXAMPLE 2: **How many ways are there to choose a subcommittee of 3 people from a committee of 5 people?**

Solution: We need to calculate "*5 choose 3*."

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!\times 2!} = \frac{120}{6\times 2} = \frac{120}{12} = \mathbf{10}$$

Is this the answer we obtained in Example 1?

We now have the tools and the talent (I stole this phrase from *Ghostbusters*) to solve the lottery problem. However, we need one more calculation example demonstrating the power of reducing a fraction before we grab our calculators.

<u>EXAMPLE 3:</u> **Calculate $\binom{10}{7}$.**

<u>Solution:</u> This is a **combination of 10 things taken 7 at a time** (and order doesn't matter).

$$\begin{aligned}\binom{10}{7} &= \frac{10!}{7!(10-7)!}\\ &= \frac{10!}{7!\,3!}\\ &= \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{[7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1]\times[3\cdot 2\cdot 1]}\\ &= \frac{10\cdot 9\cdot 8\cdot \cancel{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}}{[\cancel{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}]\times[3\cdot 2\cdot 1]}\\ &= \frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1}\\ &= \mathbf{120}\end{aligned}$$

<u>EXAMPLE 4:</u> **The Lottery Question: How many ways can 6 numbers be chosen from a set of 51 numbers?**

<u>Solution:</u> We apply the combination formula again:

$$\begin{aligned}\binom{51}{6} &= \frac{51!}{6!\,(51-6)!}\\ &= \frac{51!}{6!\times 45!}\\ &= \frac{51\times 50\times 49\times 48\times 47\times 46\times 45\times 44\times 43\times\cdots\times 3\times 2\times 1}{[6\times 5\times 4\times 3\times 2\times 1]\times[45\times 44\times 43\times\cdots\times 3\times 2\times 1]}\\ &= \frac{51\times 50\times 49\times 48\times 47\times 46\times \cancel{45\times 44\times 43\times\cdots\times 3\times 2\times 1}}{[6\times 5\times 4\times 3\times 2\times 1]\times[\cancel{45\times 44\times 43\times\cdots\times 3\times 2\times 1}]}\\ &= \frac{12{,}966{,}811{,}200}{720} = \mathbf{18{,}009{,}460}\end{aligned}$$

Since there are over 18 million ways to select 6 out of 51 numbers, you have about 1 chance in 18 million to win the lottery. And yes, you could win the lottery — guaranteed — by buying 18,009,460 different lottery tickets. What are the fallacies in this reasoning?

Our final example involves the combination formula using a variable.

EXAMPLE 5: **How many ways are there to select 2 objects out of *n* objects? (We assume $n \geq 2$.)**

Solution: The combination formula gives us

$$\binom{n}{2} = \frac{n!}{2! \times (n-2)!} = \frac{n(n-1)(n-2)(n-3)\cdots(2)(1)}{[2\times 1]\times[(n-2)(n-3)\cdots(2)(1)]}$$

$$= \frac{n(n-1)\cancel{(n-2)(n-3)\cdots(2)(1)}}{2\times \cancel{(n-2)(n-3)\cdots(2)(1)}} = \frac{n(n-1)}{2} = \boldsymbol{\frac{n^2-n}{2}}$$

Homework

1. Calculate: a. $\binom{7}{3}$ b. $\binom{10}{9}$ c. $\binom{12}{6}$ d. $\binom{20}{12}$

2. Calculate: a. $\binom{100}{98}$ b. $\binom{200}{1}$ c. $\binom{132}{0}$ d. $\binom{750}{750}$

3. Calculate: a. $\binom{n}{0}$ b. $\binom{n}{1}$ c. $\binom{n}{2}$ d. $\binom{n}{3}$

4. Calculate: a. $\binom{n}{n}$ b. $\binom{n}{n-1}$ c. $\binom{n}{n-2}$ d. $\binom{n}{n-3}$

5. a. Show that $\binom{1{,}000}{250} = \binom{1{,}000}{750}$.

b. [Hard] Now try to prove that $\binom{n}{k} = \binom{n}{n-k}$.

6. Calculate:

a. $\binom{2}{0}+\binom{2}{1}+\binom{2}{2}$

b. $\binom{3}{0}+\binom{3}{1}+\binom{3}{2}+\binom{3}{3}$

c. $\binom{4}{0}+\binom{4}{1}+\binom{4}{2}+\binom{4}{3}+\binom{4}{4}$

d. $\binom{5}{0}+\binom{5}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4}+\binom{5}{5}$

e. $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n-1}+\binom{n}{n}$

7. How many ways are there to choose 5 numbers in a lottery containing the numbers 1 through 100?

Review Problems

8. How many ways are there to choose 12 letters from the first 15 letters of the alphabet?

9. In a lottery with 75 numbers, how many ways can you choose 3 numbers?

10. How many ways can you select 3 objects from n objects?

11. Calculate: a. $\binom{197}{197}$ b. $\binom{299}{0}$ c. $\binom{1024}{1}$

12. Prove that $\binom{1000}{123} = \binom{1000}{877}$.

13. True/False:

    a. $\binom{n}{0} = \binom{n}{n}$

    b. $\binom{p}{q} = \binom{p}{p-q}$

    c. There are over 20 million ways to choose 6 numbers from a set of 51 numbers.

    d. There are $\frac{n^2-n}{2}$ ways to choose 2 people out of n people.

Solutions

1. a. 35 b. 10 c. 924 d. 125,970

2. a. 4,950 b. 200 c. 1 d. 1

3. a. 1 b. n c. $\frac{n^2-n}{2}$ d. $\frac{n^3-3n^2+2n}{6}$

4. a. 1 b. n c. $\dfrac{n^2-n}{2}$ d. $\dfrac{n^3-3n^2+2n}{6}$

5. a. $\dbinom{1000}{250} = \dfrac{1000!}{250!\times(1000-250)!} = \dfrac{1000!}{250!\times 750!}$. Also,

$\dbinom{1000}{750} = \dfrac{1000!}{750!\times(1000-750)!} = \dfrac{1000!}{750!\times 250!}$. Since these last two expressions are equivalent (no matter their final numerical values), we've proved the assertion.

b. Like I'm going to give this one away!

6. a. 4 b. 8 c. 16 d. 32
e. I'm curious what you got.

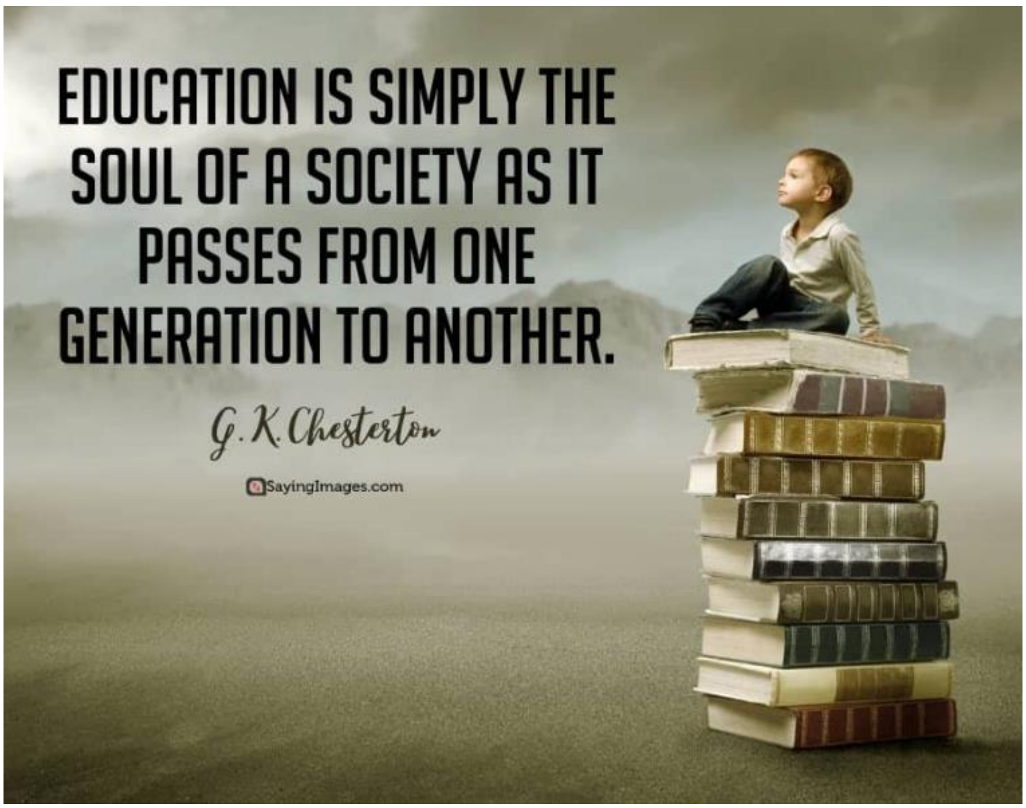
7. 75,287,520

8. 455 **9**. 67,525 **10**. $\frac{1}{6}n^3-\frac{1}{2}n^2+\frac{1}{3}n$

11. a. 1 b. 1 c. 1,024

12. Convince yourself.

13. a. T b. T c. F d. T


EDUCATION IS SIMPLY THE SOUL OF A SOCIETY AS IT PASSES FROM ONE GENERATION TO ANOTHER.
G.K. Chesterton
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